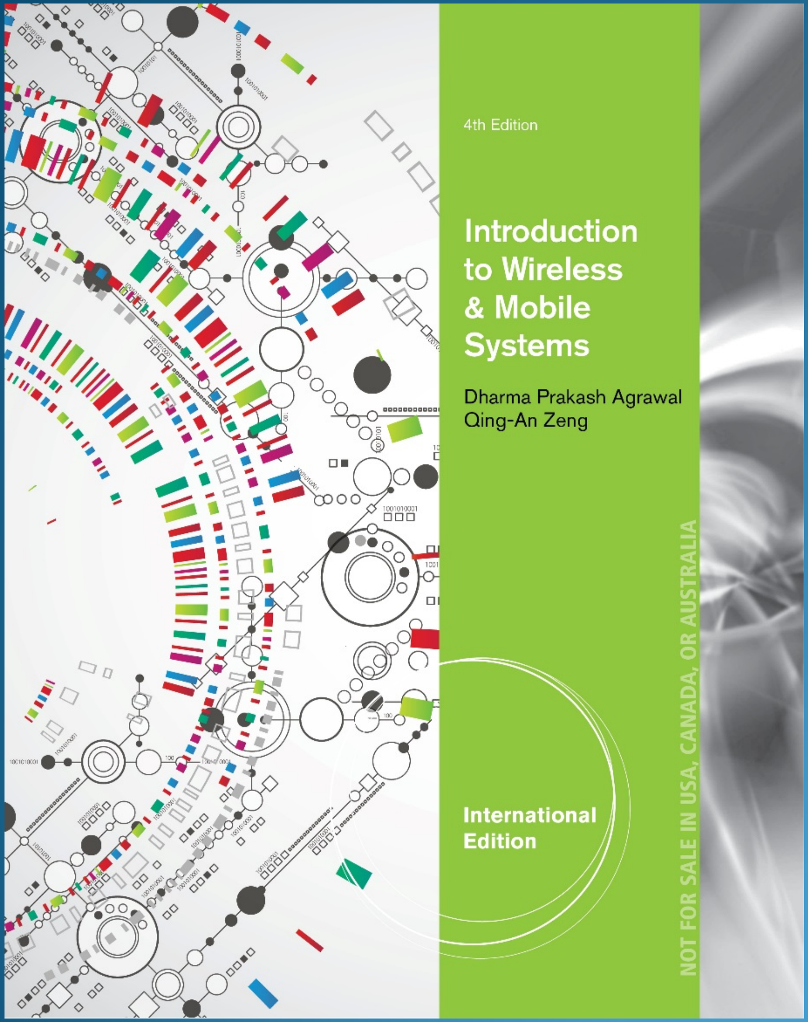


Chapter 2

Probability, Statistics, and Traffic Theories

Outline

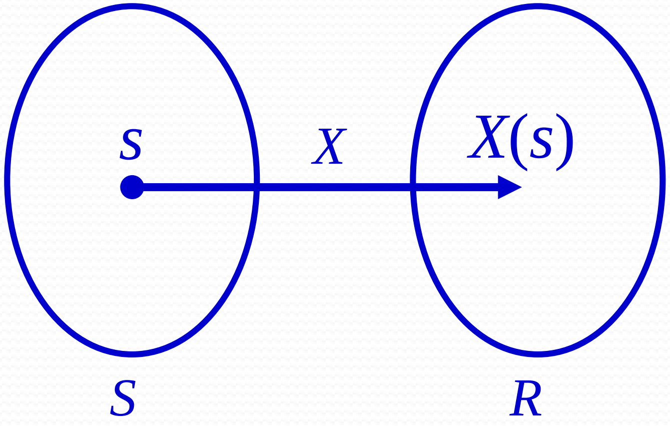
- Introduction
- Probability Theory and Statistics Theory
  - Random variables
  - Probability mass function (pmf)
  - Probability density function (pdf)
  - Cumulative distribution function (CDF)
  - Expected value, nth moment, nth central moment, and variance
  - Some important distributions
- Traffic Theory
  - Poisson arrival model, etc.
- Basic Queuing Systems
  - Little’s law
  - Basic queuing models

Introduction

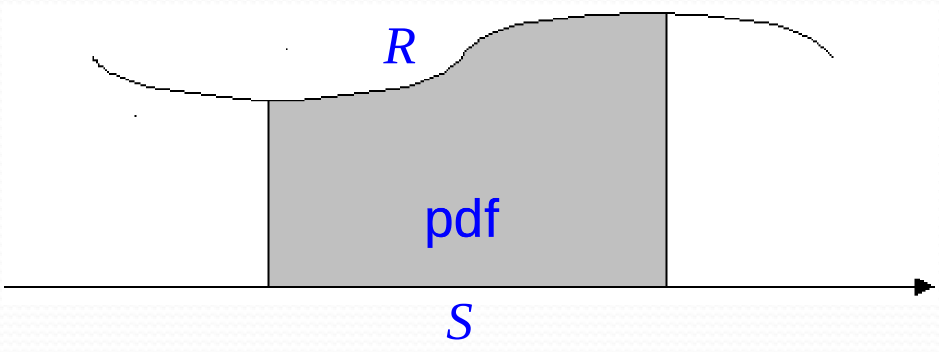
- Several factors influence the performance of wireless systems:
  - Density of mobile users
  - Cell size
  - Moving direction and speed of users (Mobility models)
  - Call rate, call duration
  - Interference, etc.
- Probability, statistics theory and traffic patterns, help make these factors tractable

Probability Theory and Statistics Theory

- Random Variables (RVs)
  - Let *S* be sample associated with experiment *E*
  - *X* is a function that associates a real number to each $s \in S$
  - RVs can be of two types: Discrete or Continuous
  - Discrete random variable => probability mass function (pmf)
  - Continuous random variable => probability density function (pdf)



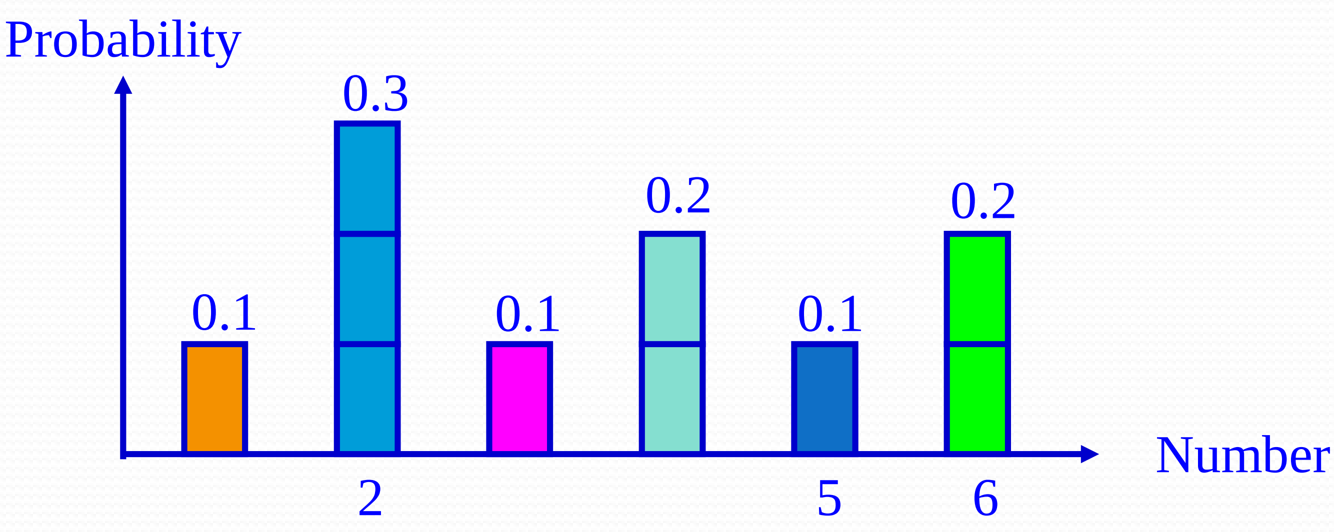
Discrete random variable



Continuous random variable

Discrete Random Variables

- In this case, *X*(*s*) contains a finite or infinite number of values
  - The possible values of *X* can be enumerated
- **For Example:** Throw a 6 sided dice and calculate the probability of a particular number appearing.



Discrete Random Variables

- The probability mass function (pmf) $p(k)$ of X is defined as:

  $p(k) = p(X = k)$, for $k = 0, 1, 2, \ldots$

  where

  1. Probability of each state occurring

     $0 \leq p(k) \leq 1$, for every k;

  2. Sum of all states

     $\sum p(k) = 1$, for all k

Continuous Random Variables

- In this case, X contains an infinite number of values
- Mathematically, X is a continuous random variable if there is a function f, called probability density function (pdf) of X that satisfies the following criteria:

    1. $f(x) \geq 0$, for all x;

    2. $\int f(x)dx = 1$

Cumulative Distribution Function

- Applies to all random variables
- A cumulative distribution function (CDF) is defined as:
  - For discrete random variables:

$$P(k) = P(X \le k) = \sum_{\text{all} \le k} P(X = k)$$

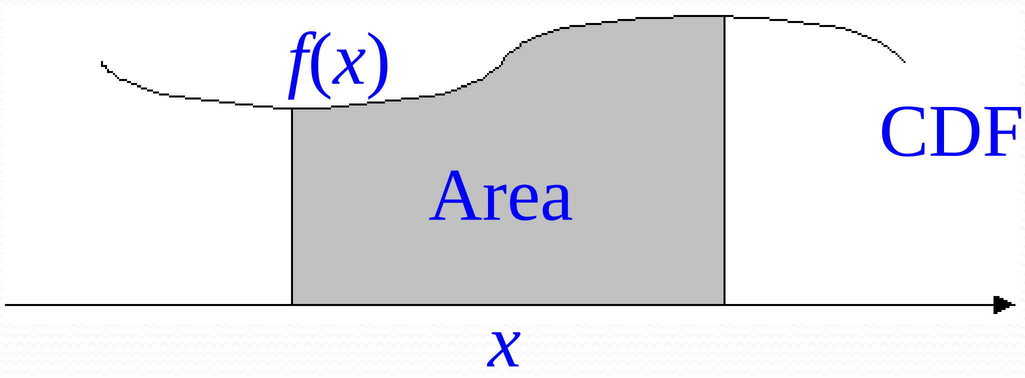
  - For continuous random variables:

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)dx$$

Probability Density Function

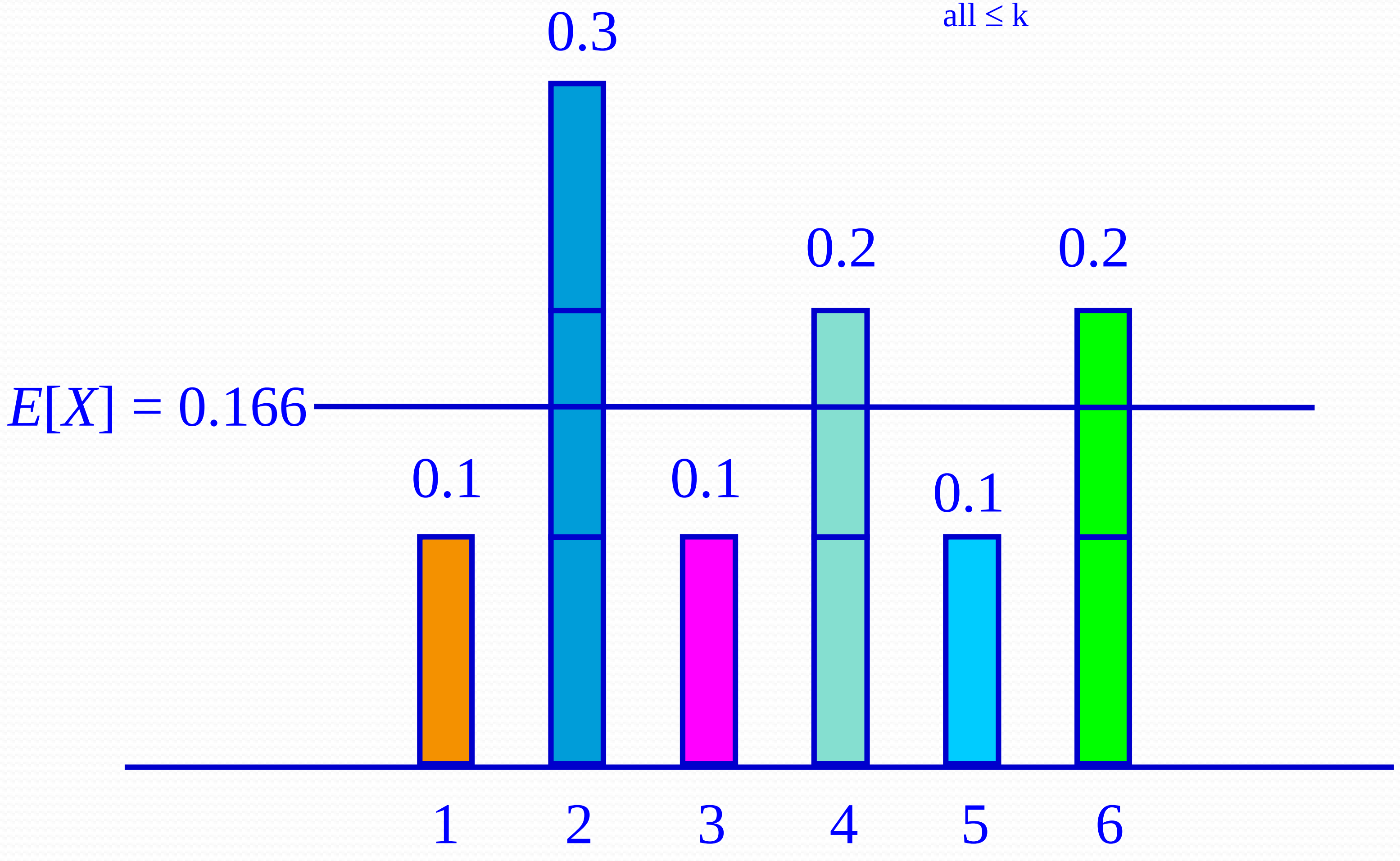
- The pdf $f(x)$ of a continuous random variable X is the derivative of the CDF $F(x)$, i.e.,

$$f(x) = \frac{dF_X(x)}{dx}$$



Expected Value, nth Moment, nth Central Moment, and Variance

Average $E[X] = \sum_{all \leq k} kP(X = k)$=1/6



Expected Value, nth Moment, nth Central Moment, and Variance

- Discrete Random Variables
  - Expected value represented by E or average of random variable

$$E[X] = \sum_{\text{all} \leq k} kP(X = k)$$

  - Variance or the second central moment

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

  - nth moment

$$E[X^n] = \sum_{\text{all} \leq k} k^n P(X = k)$$

  - nth central moment

$$E[(X - E[X])^n] = \sum_{\text{all} \leq k} (k - E[X])^n P(X = k)$$

Expected Value, nth Moment, nth Central Moment, and Variance

- Continuous Random Variable
  - Expected value or mean value

$$E[X] = \int_{-\infty}^{+\infty} x f(x) dx$$

  - Variance or the second central moment

$$\sigma^2 = Var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

  - n^{th} moment

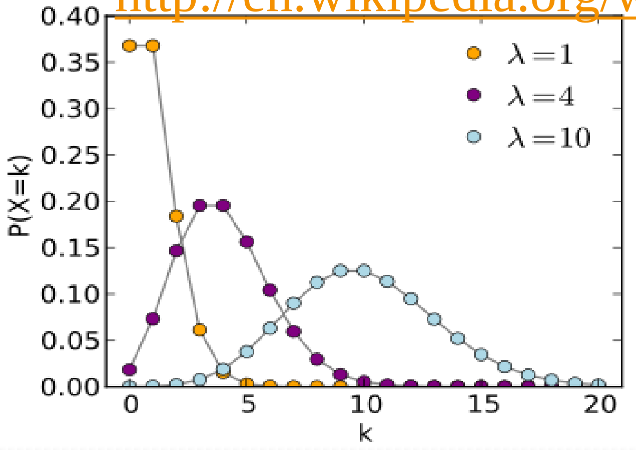
$$E[X^n] = \int_{-\infty}^{+\infty} x^n f(x) dx$$

  - n^{th} central moment

$$E[(X - E[X])^n] = \int_{-\infty}^{+\infty} (x - E[X])^n f(x) dx$$

Some Important Discrete Random Distributions

http://en.wikipedia.org/wiki/



- Poisson

$$P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!},\ k=0,1,2,\ldots,\text{ and } \lambda>0$$

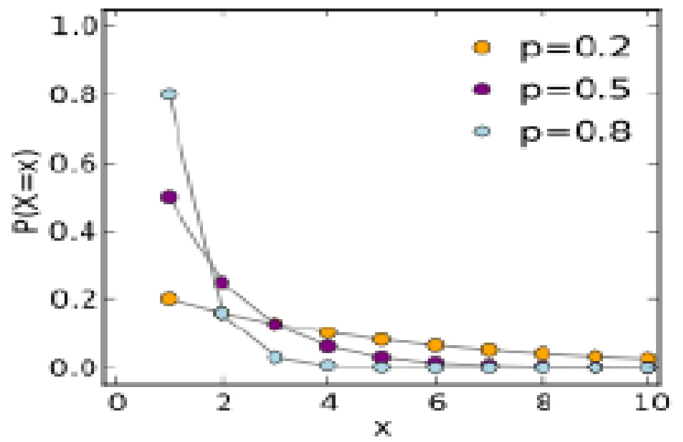
  - $E[X] = \lambda$, and $Var(X) = \lambda$

- Geometric

$$P(X = k) = p(1-p)^{k-1} \ ,$$

where p is success probability

  - $E[X] = 1/(1-p)$, and $Var(X) = p/(1-p)^2$



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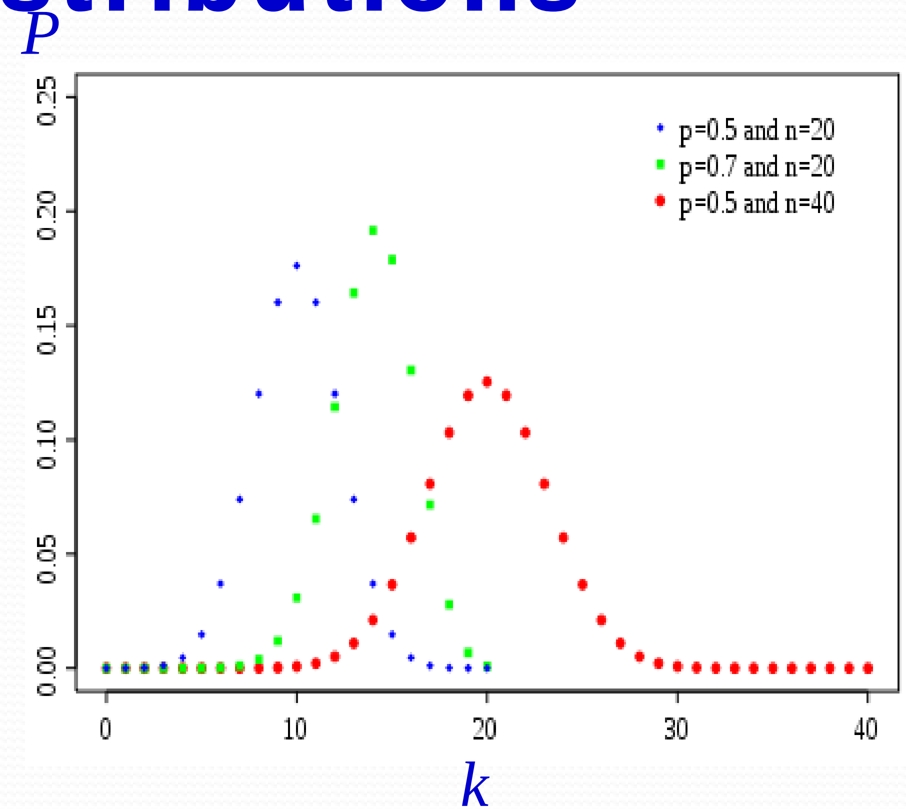
Some Important Discrete Random Distributions

- Binomial

  Out of *n* dice, exactly *k* dice have the same value: probability p^k and (*n*-*k*) dice have different values: probability$(1-p)^{n-k}$.

  For any *k* dice out of *n*:



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$$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k},$$

where,

$k=0,1,2,\ldots,n$; $n=0,1,2,\ldots$; p is the sucess probability, and

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

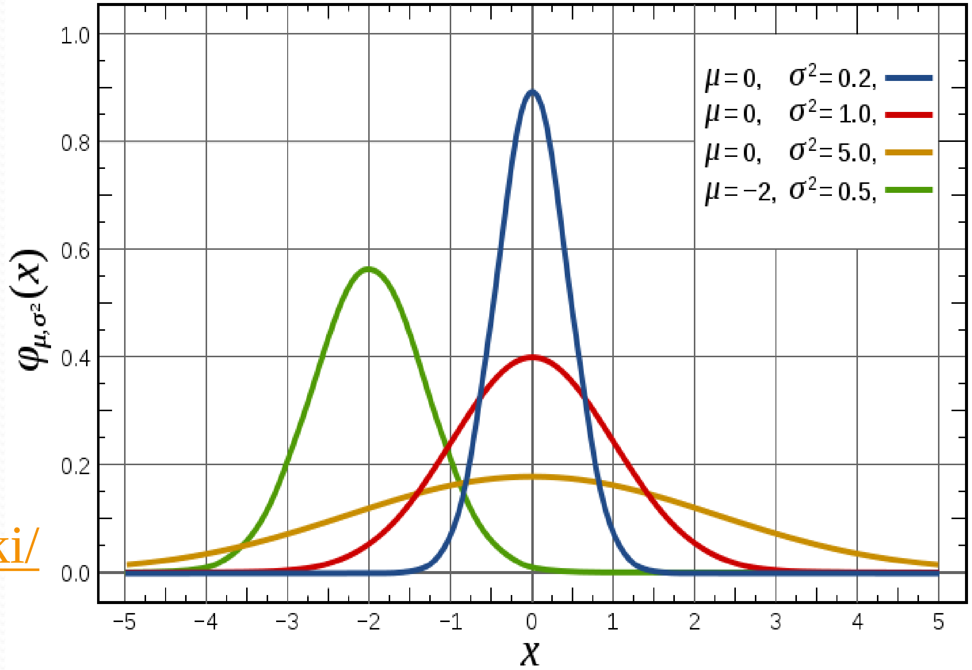
Some Important Continuous Random Distributions

- Normal: $E[X] = \mu$, and $Var(X) = \sigma^2$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}}, \text{ for } -\infty < x < \infty$$

and the cumulative distribution function can be obtained by

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{\frac{-(y-\mu)^2}{2\sigma^2}} dy$$



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Some Important Continuous Random Distributions

- Uniform

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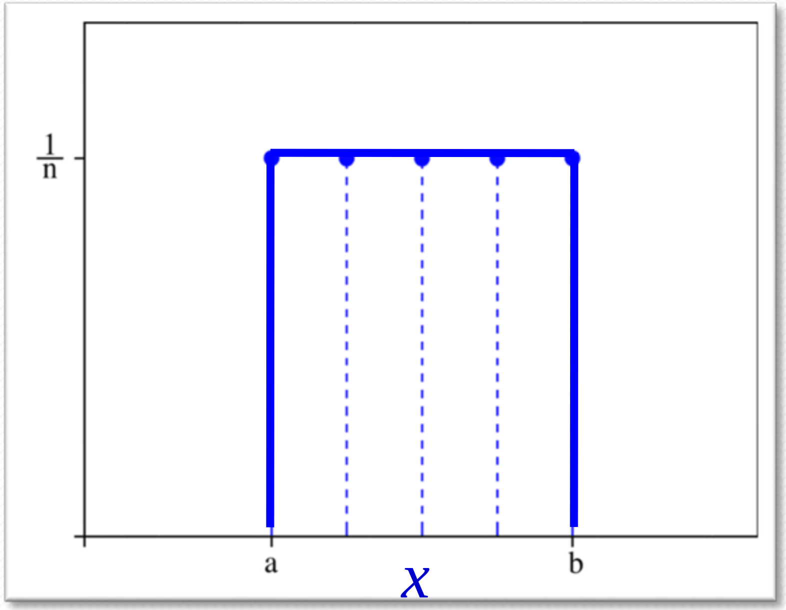


$$f_X(x) = \begin{cases} \dfrac{d}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$

and the cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & \text{for } x < a \\ \dfrac{x-a}{b-a}, & \text{for } a \le x \le b \\ 1, & \text{for x > b} \end{cases}$$

- $E[X] = (a+b)/2$, and $Var(X) = (b\text{-}a)^2/12$

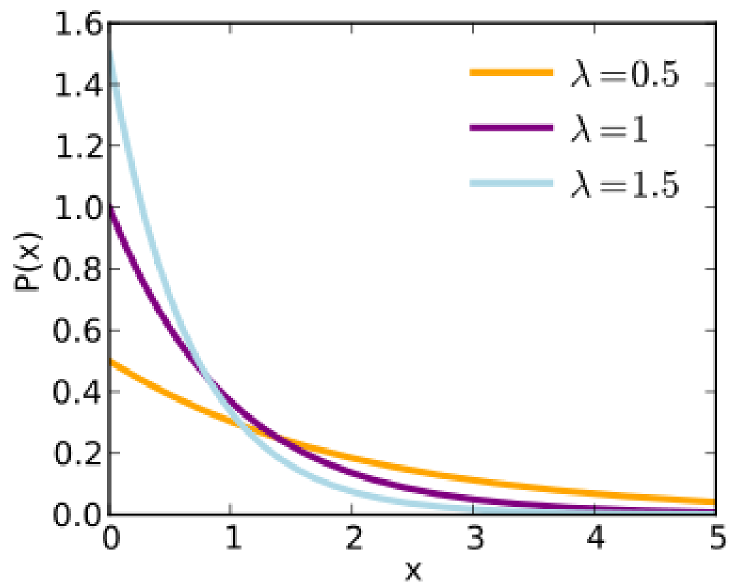
Some Important Continuous Random Distributions

- Exponential

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \lambda e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$

and the cumulative distribution function is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x}, & \text{for } 0 \le x < \infty \end{cases}$$



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- *E*[*X*] = *1/λ*, and *Var*(*X*) = *1/λ²*

Multiple Random Variables

- There are cases where the result of one experiment determines the values of several random variables
- The joint probabilities of these variables are:
  - Discrete variables:

    $p(x_1, \ldots, x_n) = P(X_1 = x_1, \ldots, X_n = x_n)$
  - Continuous variables:

    CDF: $F_{x1x2\ldots xn}(x_1, \ldots, x_n) = P(X_1 \le x_1, \ldots, X_n \le x_n)$

    pdf: $$f_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n) = \frac{\partial^n F_{X_1, X_2, \ldots X_n}(x_1, x_2, \ldots x_n)}{\partial x_1 \partial x_2 \ldots \partial x_n}$$

Independence and Conditional Probability

- Independence: The random variables are said to be independent of each other when the occurrence of one does not affect the other. The pmf for discrete random variables in such a case is given by:

  $p(x_1,x_2,\ldots x_n)=P(X_1=x_1)P(X_2=x_2)\ldots P(X_3=x_3)$ and for continuous random variables as:

  $F_{X1,X2,\ldots Xn} = F_{X1}(x_1)F_{X2}(x_2)\ldots F_{Xn}(x_n)$

- Conditional probability: is the probability that $X_1= x_1$ given that $X_2= x_2$. Then for discrete random variables the probability becomes:

$$P(X_1 = x_1 \mid X_2 = x_2,\ldots, X_n = x_n) = \frac{P(X_1 = x_1, X_2 = x_2,\ldots, X_n = x_n)}{P(X_2 = x_2,\ldots, X_n = x_n)}$$

and for continuous random variables it is:

$$P(X_1 \le x_1 \mid X_2 \le x_2,\ldots, X_n \le x_n) = \frac{P(X_1 \le x_1, X_2 \le x_2,\ldots, X_n \le x_n)}{P(X_2 \le x_2,\ldots X_n \le x_n)}$$

Bayes Theorem

- A theorem concerning conditional probabilities of the form $P(X|Y)$ (read: the probability of X, given Y) is

$$P(X \mid Y) = \frac{P(Y \mid X)P(X)}{P(Y)}$$

where $P(X)$ and $P(Y)$ are the unconditional probabilities of X and Y, respectively

Important Properties of Random Variables

- Sum property of the expected value
  - Expected value of the sum of random variables:

$$E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]$$

- Product property of the expected value
  - Expected value of product of stochastically independent random variables

$$E\left[\prod_{i=1}^{n} X_i\right] = \prod_{i=1}^{n} E[X_i]$$

Important Properties of Random Variables

- Sum property of the variance
  - Variance of the sum of random variables is

$$Var\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} a_i a_j \operatorname{cov}[X_i, X_j]$$

where $cov[X_i,X_j]$ is the covariance of random variables X_i and X_j and

$$\begin{aligned}\operatorname{cov}[X_i, X_j] &= E[(X_i - E[X_i])(X_j - E[X_j])] \\ &= E[X_i X_j] - E[X_i]E[X_j]\end{aligned}$$

If random variables are independent of each other, i.e., $cov[X_i,X_j]=0$, then

$$Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i)$$

Important Properties of Random Variables

- **Distribution of sum -** For continuous random variables with joint pdf $f_{XY}(x, y)$ and if $Z = \Phi(X,Y)$, the distribution of Z may be written as

$$F_Z(z) = P(Z \leq z) = \int_{\phi Z} f_{XY}(x, y)dxdy$$

where Φ_Z *is a subset of* Z.

- For a special case $Z= X+Y$

$$Fz(z) = \iint_{\phi Z} f_{XY}(x, y)dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy$$

- If X and Y are independent variables, the $f_{XY}(x,y)= f_X(x)f_Y(y)$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)dx, \quad \text{for } -\infty \leq z < \infty$$

- If both X and Y are non negative random variables, then pdf is the convolution of the individual pdfs, $f_X(x)$ and $f_Y(y)$

$$f_Z(z) = \int_{0}^{z} f_X(x) f_Y(z - x)dx, \quad \text{for } -\infty \leq z < \infty$$

Central Limit Theorem

The *Central Limit Theorem* states that whenever a random sample (X_1, X_2,.. X_n) of size *n* is taken from any distribution with expected value $E[X_i] = \mu$ and variance $Var(X_i) = \sigma^2$, where *i* =1,2,..,n, then their arithmetic mean is defined by

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Central Limit Theorem

- Mean of a sufficiently large number of independent random variables, each with finite mean and variance, will be approximately normally distributed
- The sample mean is approximated to a normal distribution with
  - $E[S_n] = \mu$, and
  - $Var(S_n) = \sigma^2 / n$
- The larger the value of the sample size *n*, the better the approximation to the normal
- This is very useful when inference between signals needs to be considered

Poisson Arrival Model

- Events occur continuously and independently of one another
- A Poisson process is a sequence of events "randomly spaced in time"
- For example, customers arriving at a bank and Geiger counter clicks are similar to packets arriving at a buffer
- The rate λ of a Poisson process is the average number of events per unit time (over a long time)

Properties of a Poisson Process

- Properties of a Poisson process
  - For a time interval $[0, t]$, the probability of n arrivals in t units of time is

$$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

  - For two disjoint (non overlapping) intervals (t_1, t_2) and (t_3, t_4), (i.e. , $t_1 < t_2 < t_3 < t_4$), the number of arrivals in (t_1, t_2) is independent of arrivals in (t_3, t_4)

Interarrival Times of Poisson Process

- Interarrival times of a Poisson process
  - We pick an arbitrary starting point t_0 in time . Let T_1 be the time until the next arrival. We have

$$P(T_1 > t) = P_0(t) = e^{-\lambda t}$$

  - Thus the cumulative distribution function of T_1 is given by

$$F_{T1}(t) = P(T_1 \leq t) = 1 - e^{-\lambda t}$$

  - The pdf of T_1 is given by

$$f_{T1}(t) = \lambda e^{-\lambda t}$$

Therefore, T_1 has an exponential distribution with mean rate λ

Exponential Distribution

- Similarly T_2 is the time between first and second arrivals, we define T_3 as the time between the second and third arrivals, T_4 as the time between the third and fourth arrivals and so on
- The random variables T_1, T_2, T_3… are called the interarrival times of the Poisson process
- T_1, T_2, T_3,… are independent of each other and each has the same exponential distribution with mean arrival rate λ

Memoryless and Merging Properties

- Memoryless property
  - A random variable *X* has the property that "the future is independent of the past" i.e., the fact that it hasn't happened yet, tells us nothing about how much longer it will take before it does happen
- Merging property
  - If we merge *n* Poisson processes with distributions for the inter arrival times

$$1 - e^{-\lambda i t} \quad \text{for } i = 1, 2, \ldots, n$$

into one single process, then the result is a Poisson process for which the inter arrival times have the distribution $1 - e^{-\lambda t}$ with mean

$$\lambda = \lambda_1 + \lambda_2 + .. + \lambda_n$$

Basic Queuing Systems

- What is queuing theory?
  - Queuing theory is the study of queues (sometimes called waiting lines)
  - Can be used to describe real world queues, or more abstract queues, found in many branches of computer science, such as operating systems
- Basic queuing theory

  Queuing theory is divided into 3 main sections:
  - Traffic flow
  - Scheduling
  - Facility design and employee allocation

Kendall's Notation

- D.G. Kendall in 1951 proposed a standard notation for classifying queuing systems into different types. Accordingly the systems were described by the notation A/B/C/D/E where:

A	Distribution of inter arrival times of customers
B	Distribution of service times
C	Number of servers
D	Maximum number of customers in the system
E	Calling population size

Kendall's notation

A and B can take any of the following distributions types:

M	Exponential distribution (Markovian)
D	Degenerate (or deterministic) distribution
E_k	Erlang distribution (k = shape parameter)
H_k	Hyper exponential with parameter k

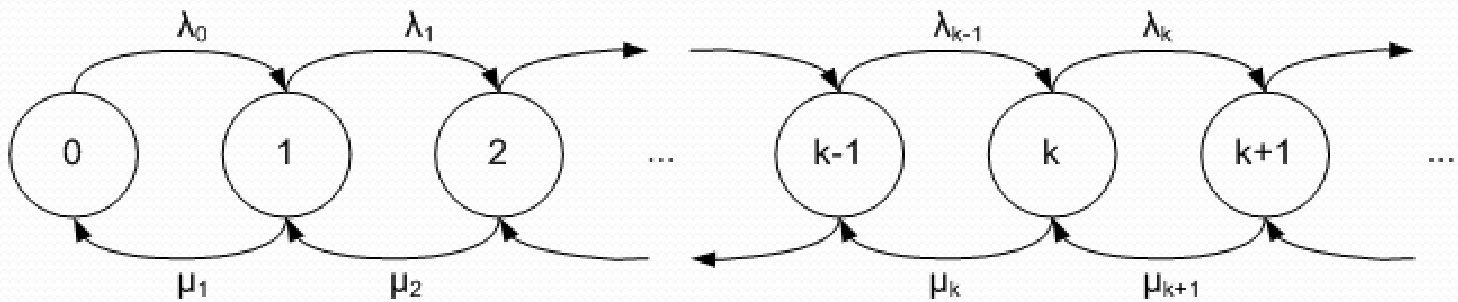
Little's Law

- Assuming a queuing environment to be operating in a stable steady state where all initial transients have vanished, the key parameters characterizing the system are:
  - λ – the mean steady state consumer arrival
  - N – the average no. of customers in the system
  - T – the mean time spent by each customer in the system

which gives

$$N = \lambda T$$

Markov Process

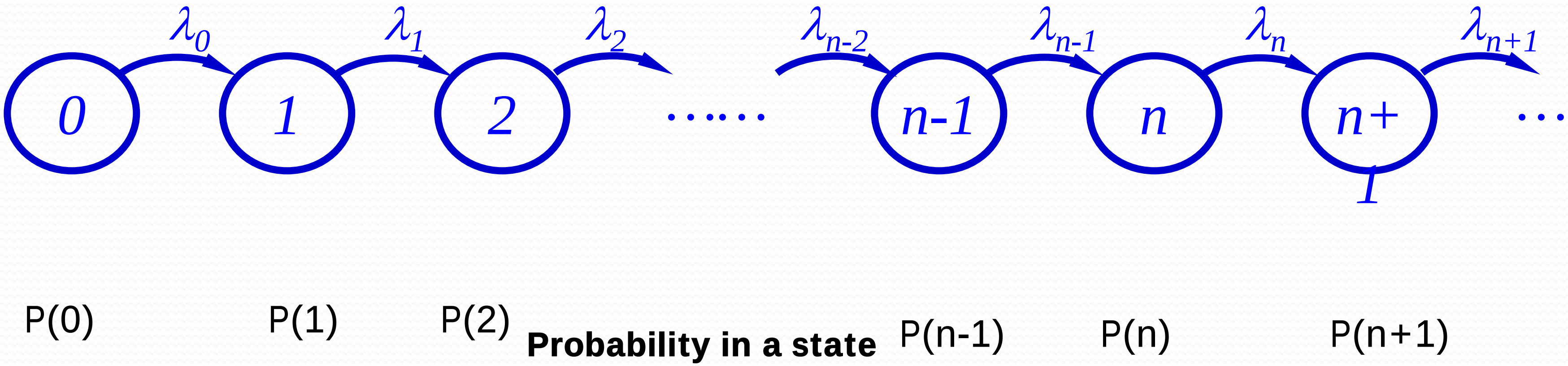


- A Markov process is one in which the next state of the process depends only on the present state, irrespective of any previous states taken by the process
- The knowledge of the current state and the transition probabilities from this state allows us to predict the next state

Birth-Death Process

- Special type of Markov process
- Often used to model a population (or, number of jobs in a queue)
- If, at some time, the population has *n* entities (*n* jobs in a queue), then birth of another entity (arrival of another job) causes the state to change to $n+1$
- On the other hand, a death (a job removed from the queue for service) would cause the state to change to $n-1$
- Any state transitions can be made only to one of the two neighboring states

State Transition Diagram

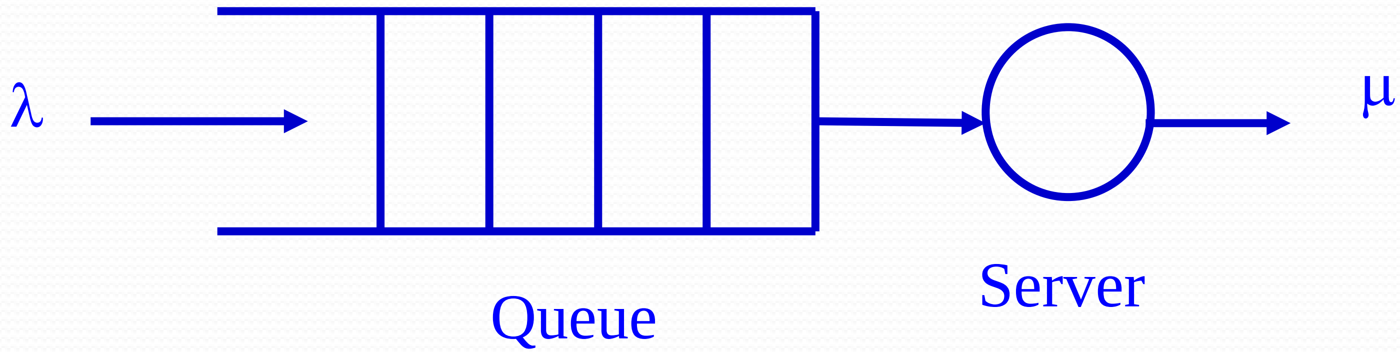


The state transition diagram of the continuous birth-death process

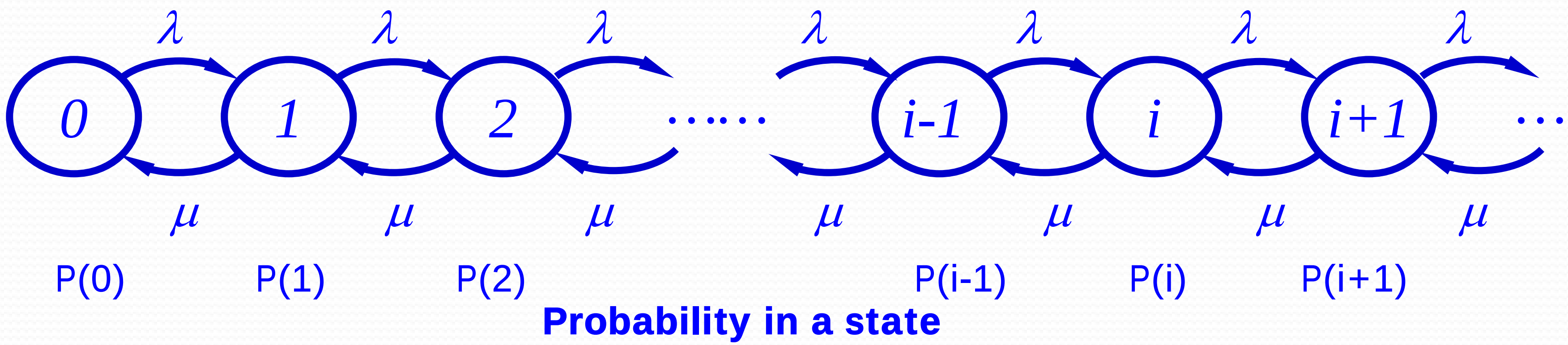
M / M / 1 / ∞ or M / M / 1 Queuing System

- Distribution of inter arrival times of customers: M
- Distribution of service times: M
- Number of servers: 1
- Maximum number of customers in the system: ∞
- When a customer arrives in this system it will be served if the server is free, otherwise the customer is queued
- In this system, customers arrive according to a Poisson distribution and compete for the service in a FIFO (first in first out) manner
- Service times are independent identically distributed (IID) random variables, the common distribution being exponential

Queuing Model and State Transition Diagram



The M/M/1/∞ queuing model



The state transition diagram of the M/M/1/∞ queuing system

Equilibrium State Equations

- If mean arrival rate is λ and mean service rate is μ, i = 0, 1, 2 be the number of customers in the system and $P(i)$ be the state probability of the system having i customers
- From the state transition diagram, the equilibrium state equations are given by

$$\lambda P(0) = \mu P(1), \qquad i = 0,$$

$$(\lambda + \mu)P(i) = \lambda P(i-1) + \mu P(i+1), \qquad i \geq 1$$

$$P(i) = \left(\frac{\lambda}{\mu}\right)^i P(0), \qquad i \geq 1$$

Traffic Intensity

- We know that the *P*(*0*) is the probability of server being free. Since *P*(*0*) > 0, the necessary condition for a system being in steady state is,

$$\rho = \frac{\lambda}{\mu} < 1$$

This means that the arrival rate cannot be more than the service rate, otherwise an infinite queue will form and jobs will experience infinite service time

Queuing System Metrics

- $\rho = 1 - P(0)$, is the probability of the server being busy. Therefore, we have

$$P(i) = \rho^i(1 - \rho)$$

- The average number of customers in the system is

$$L_s = \frac{\lambda}{\mu - \lambda}$$

- The average dwell time of customers is

$$W_s = \frac{1}{\mu - \lambda}$$

Queuing System Metrics

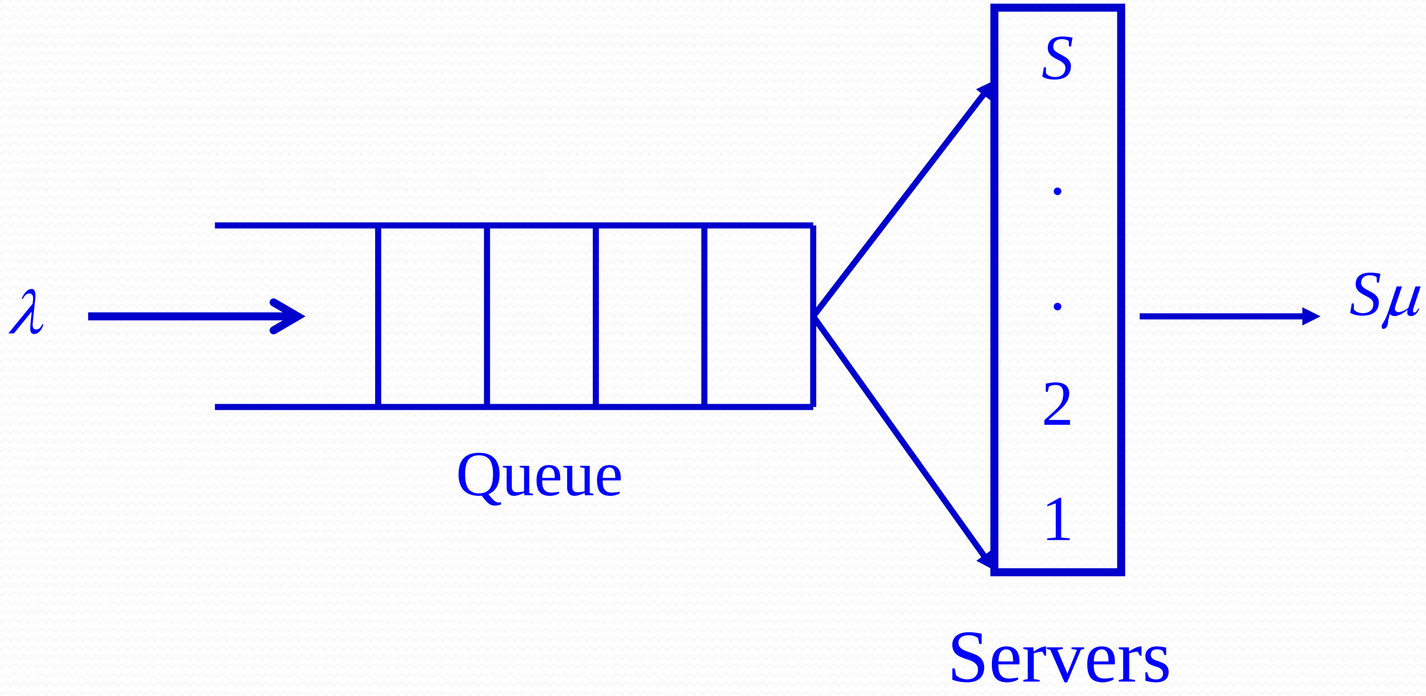
- The average queuing length is

$$Lq = \sum_{i=1}^{\infty} (i-1)P(i) = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}$$

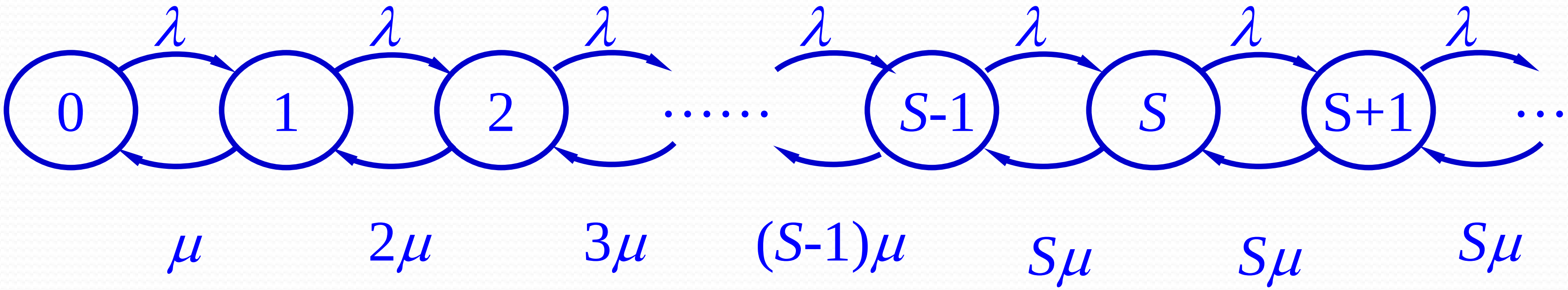
- The average waiting time of customers is

$$W_q = \frac{L_q}{\lambda} = \frac{\rho^2}{\lambda(1-\rho)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

M/M/S/∞ Queuing Model



State Transition Diagram



Queuing System Metrics

- The average number of customers in the system is

$$L_S = \sum_{i=0}^{\infty} iP(i) = \alpha + \frac{\rho \alpha^S P(0)}{S!(1-\rho)^2}$$

- The average dwell time of a customer in the system is given by

$$W_S = \frac{L_S}{\lambda} = \frac{1}{\mu} + \frac{\alpha^S P(0)}{S\mu . S!(1-\rho)^2}$$

Queuing System Metrics

- The average queue length is

$$L_q = \sum_{i=s}^{\infty} (i - S) P(i) = \frac{\alpha^{S+1} P(0)}{(S-1)!(S-\alpha)^2}$$

- The average waiting time of customers is

$$W_q = \frac{L_q}{\lambda} = \frac{\alpha^S P(0)}{S\mu . S!(1-\rho)^2}$$

M/G/1/∞ Queuing Model

- We consider a single server queuing system whose arrival process is Poisson with mean arrival rate λ
- Service times are independent and identically distributed with distribution function F_B and pdf f_b
- Jobs are scheduled for service as FIFO

Basic Queuing Model

- Let $N(t)$ denote the number of jobs in the system (those in queue plus in service) at time t.
- Let t_n (n= 1, 2,..) be the time of departure of the n^{th} job and X_n be the number of jobs in the system at time t_n, so that

$$X_n = N(t_n), \quad \text{for } n = 1, 2, ..$$

- The stochastic process can be modeled as a discrete Markov chain known as imbedded Markov chain, which helps convert a non-Markovian problem into a Markovian one.

Queuing System Metrics

- The average number of jobs in the system, in the steady state is

$$E[N] = \rho + \frac{\lambda^2 E[B^2]}{2(1-\rho)}$$

- The average dwell time of customers in the system is

$$Ws = \frac{E[N]}{\lambda} = \frac{1}{\mu} + \frac{\lambda E[B^2]}{2(1-\rho)}$$

- The average waiting time of customers in the queue is

$$E[N] = \lambda W_q + \rho$$

- Average waiting time of customers in the queue is

$$W_q = \frac{\lambda E[B^2]}{2(1-\rho)}$$

- The average queue length is

$$L_q = \frac{\lambda^2 E[B^2]}{2(1-\rho)}$$